A First Look at Combinatorial Species

Saiyue Lyu(20654752)

The theory of combinatorial species, first introduced by André Joyal in 1981, is a unified algebraic method to understanding combinatorial structures in terms of generating functions, which allows people to transfer recursive definitions of tree-like structures into functional equations, thus count the number of labelled and unlabelled structures that share some properties. This report gives an overview of basic concepts of the combinatorial theory of species from the perspective of category theory.

1. Motivation

Species are related to generating functions very much and they fundamentally describe labelled objects. One of the most famous example to see this transformation from recursive definition to functional equation is the Catalan Number, which is constructed as follows:

Let $\mathcal{T}$ be the set of binary rooted ordered trees. Each tree decomposes as the root vertex and an ordered pair of binary trees, namely, the left branch and the right branch. Hence we obtain a recursive decomposition of $\mathcal{T}$,which is a recursive structure bijection:

$$\mathcal{T} \simeq \{\bullet\} \times (\mathcal{T} \cup \epsilon) \times (\mathcal{T} \cup \epsilon)$$

where $\bullet$ represents the root, ϵ represents empty, thus $(\mathcal{T} \cup \epsilon)$ represents left branch or right branch. Define the weight function $w : \mathcal{T} \to \mathbb{Z}$ to behave as $w(t) = 1 + w(t_1) + w(t_2)$ whenever $t \mapsto (\bullet, t_1, t_2)$. Then we think about the generating series (which will be defined formally later):

$$\begin{aligned} T(x) &= \sum_{t\in\mathcal{T}} x^{w(t)} \\ &= \sum_{(\bullet,t_1,t_2)\in\{\bullet\}\times(\mathcal{T}\cup\epsilon)\times(\mathcal{T}\cup\epsilon)} x^{1+w(t_1)+w(t_2)} \\ &= (x^1)\Big(\sum_{t_1\in\mathcal{T}\cup\{\epsilon\}} x^{w(t_1)}\Big)\Big(\sum_{t_2\in\mathcal{T}\cup\{\epsilon\}} x^{w(t_2)}\Big) \\ &= x\big(T(x)+1\big)^2 \end{aligned}$$

which is a quadratic equation for $T(x)$. Solving it using quadratic formula to get:

$$T(x) = \frac{1 - 2x \pm (1-4x)^{\frac{1}{2}}}{2x}$$

Using generalized binomial theorem:

$$(1-4x)^{\frac{1}{2}} = 1 + \sum_{k\geq 1} -2^k \frac{(2k-2)!}{k!\,2^{k-1}\,(k-1)!} x^k = 1 + \sum_{k\geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k$$

Therefore we have:

$$T(x) = \textstyle\sum_{n\geq 1} \frac{1}{n+1}\binom{2n}{n} x^n$$

where $\frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan Number, which is also the number of these binary rooted trees of size n.

From this example, we see the importance of generating functions while it is difficult to count the number of combinatorial structures directly. To generalize this way of enumeration, Joyal formulated the theory of species by taking the notion of labelled structures as fundamental and build other notions (unlabelled, asymmetric, weighted, etc) based on it. More details will be provided later to show that unlabelled structure correspond to the equivalence classes of labelled structures.

But why do we need to label the nodes? In the tree shown in Figure 1, each node is uniquely determined by a path from the root. In these cases, it is not necessary to use labels. But consider the oriented cycle below, the structure has five-fold rotational symmetry, which means in this case it is impossible to uniquely refer to a node without labelling.

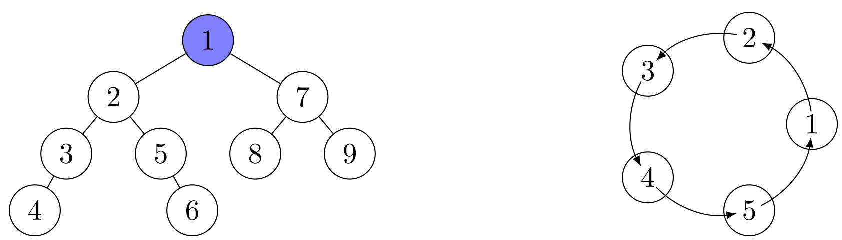

Figure 1: Two kinds of Labelled Structures

With the notion of label and the great help of generating function, it is easier to analysis and enumerate these structures using some general algebraic tools.

2. Basic Definitions

Definition 2.1. *A structure $s = (\gamma, U)$ is a construction γ performing on a finite set U*

We usually say that U is the underlying set of the structure s and in turn s is a structure labeled by the set U. It may be abstract to understand the definition, so we consider an example: the binary tree in Figure 1, it achieves a structure $s_1 = (\gamma_1, U_1)$, where

$$
\begin{aligned}
U_1 &= \{1,2,3,4,5,6,7,8,9\} \\
\gamma_1 &= \Big(\{1\}, \big\{\{1,2\}, \{1,7\}, \{2,3\}, \{2,5\}, \{3,4\}, \{5,6\}, \{7,8\}, \{7,9\}\big\}\Big)
\end{aligned}
$$

We can see that the structure consists of the root and all edges of the tree. Notice $\{1\}$ appears separately, which distinguishes from non-rooted structures. For example, the oriented cycle in Figure 1, it achieves a structure $s_2 = (\gamma_2, U_2)$, where

$$
\begin{aligned}
U_2 &= \{1,2,3,4,5\} \\
\gamma_2 &= \{(1,2), (2,3), (3,4), (4,5), (5,1)\}
\end{aligned}
$$

Another example of structure is a-subset-of-a-set, where

$$U_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$\gamma_3 = \{1, 2, 3, 4, 5\}$$

One can see that it is impossible to recover U_3 only using γ_3. So we must be careful of the abuse of notations $s = \gamma$ since this may cause ambiguity as in this example.

We could understand a structure s determined by both U and γ via axiomatic set theoretic approach. To see the connection to Category Theory, we use a functional approach involving isomorphisms. Intuitively, we can think about the previous example of rooted tree. Suppose we have a rooted tree $G = (r, V, E)$, where r is the root, V is the vertx set and E is the edge set. The structure $s = (\gamma, U)$ is obtained such that $\gamma = \big(\{r\}, E\big)$ and $U = V$. We replace each elements in U by $U' = V'$ via a bijection $\sigma : V \to V'$, which gives a new rooted tree $G' = (\sigma(r), V', E')$. We say these two graphs G and G' are isomorphic, denote as $G' = \sigma \cdot G$, where σ is the isomorphism of G to G'. This is an example to see the transport of structures along bijections. The general form of the above rooted tree is represented by the unlabelled tree :

Figure 2: Labelled Structures Revisited

Moreover, when the set U and U' coincide, we could see that σ is an automorphism of U onto itself, i.e. $G = \sigma \cdot G$. Below illustrates a rooted tree automorphism:

Figure 3: Example of Rooted Tree Automorphism

To generalize the idea from graphs, given two finite sets U and U', we want an isomorphism $\sigma : U \to U'$ so that we can transport structures between $s = (\gamma, U)$ and $s' = (\gamma', U')$. This reveals a category:

Theorem 2.2. *The collection of $s = (\gamma, U)$ corresponds to a category $\mathcal{F}$, where the morphism is just $\sigma : U \to U'$. I.e. a category of finite sets and funcctions.*

Proof. The composition law is well-defined by above construction:

$$\sigma \circ (\delta \circ \tau) = (\sigma \circ \delta) \circ \tau$$

where $\sigma \in Hom_{\mathcal{C}}\big(U_1, U_2\big)$, $\delta \in Hom_{\mathcal{C}}\big(U_2, U_3\big)$ and $\tau \in Hom_{\mathcal{C}}\big(U_3, U_4\big)$
Also the identity map is the map assigns same structure under the same finite set, i.e. the identity map $id_U : U \to U$ satisfies

$$id_U \circ \sigma = \sigma \ , \ \delta \circ id_U = \delta$$

Therefore this forms a category of finite sets. ∎

This category is called **F**inset. Moreover we can consider the core of the **F**inset, thus the category of finite sets and bijections(permutations).In section 4, we will see it is equivalent to the permutation category $\mathbb{P}$.

With $\mathcal{F}$ being a category, it is natural to see how we can find a corresponding functor. Consider the functor $F : \mathcal{F} \to \mathcal{F}$. For each finite set U, we denote by $F[U]$ the set of all structures labelled by U, i.e.

$$F[U] = \{s \,|\, s = (\gamma, U)\}$$

By definition, $F[U]$ is a finite set. Moreover, by transport of structure, each bijection $\sigma : U \to U'$ induces a bijection:

$$F[\sigma] : F[U] \to F[U'] \text{ via } s = (\gamma,\, U) \mapsto \sigma \cdot s = (\sigma \cdot \gamma,\, U')$$

Above construction gives rise to the definition of species.

Definition 2.3. *A species is a functor F from above category $\mathcal{F}$ with bijections to itself. To be more detailed,*

(1) F assigns each finite set U to a finite set $F[U]$

(2) F assigns each bijection of finite sets $\sigma : U \to U'$ to
a bijection of finite sets $F[\sigma] : F[U] \to F[U']$

such that

- *for all bijections $\sigma : U_1 \to U_2$ and $\tau : U_2 \to U_3$,*

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$$

- *for the identity map $id_U : U \to U$,*

$$F[id_U] = id_{F[U]}$$

An element $s \in F[U]$ is called an F-structure on U. The element $F[\sigma]$ is called the transport of F-structure along σ.

Recall the definition of Groupoids in Assignment 1, a species is actually a presheaf on the permutation groupoid. Later in section 4 we will see a species is equivalent to a symmetric sequence categorically.

The functionality guarantees $F[\sigma]$ to be a bijection. And $F[U]$ is independent of the nature of the elements of U by an invariance under relabeling. In this way the concepts of species of structures lay as much emphasis on isomorphisms as on the structures themselves.

Definition 2.4. *For two F-structures $s_1, s_2 \in F[U]$. A bijection $\sigma : U_1 \to U_2$ is called an isomorphism of s_1 to s_2 if $s_2 = \sigma \cdot s_1 = F[\sigma](s_1)$. And s_1, s_2 are said to have the same isomorphism type. Moreover, an isomorphism from s to s is an automorphism of s.*

There are many common examples of species :

• The empty species $\mathbf{0}$, which assigns each finite set U to $\varnothing$.

• The species $\mathbf{E}$ of set, which assigns each finite set U to the singleton $\mathbf{E}[U] = \{U\}$. If $\sigma \in Hom(U, U')$, then $\mathbf{E}[\sigma](U) = U'$. The set U can be visualized as a collection of dots labelled with elements of U.

• The species $\mathcal{E}$ of set, which assigns each finite set U to the singleton $\mathcal{E}[U] = U$. The structures on U are the elements of U.

• The species $\mathbf{1}$ of characteristic of empty set, which assigns a finite set U to

$$\mathbf{1}[U] = \begin{cases} \{U\}, & \text{if } |U| = \varnothing \\ \varnothing, & \text{otherwise} \end{cases}$$

• The species $\mathbf{X}$ of characteristic of singleton, which assigns a finite set U to

$$\mathbf{X}[U] = \begin{cases} \{U\}, & \text{if } |X| = 1 \\ \varnothing, & \text{otherwise} \end{cases}$$

• More generally, the species $\mathbf{E_m}$ of $m-$Sets, which assigns a finite set U to

$$\mathbf{E_m}[U] = \begin{cases} \{U\}, & \text{if } |U| = m \\ \varnothing, & \text{otherwise} \end{cases}$$

If $\sigma : U \to U'$ is a bijection, it forces $|U| = |U'| = m$

• the species $\mathcal{G}$ of simple graphs,which assigns a finite vertex set U to the set of simple graphs on the labels in U, i.e.

$$\mathcal{G}[U] = \left\{ G = (U, E) \,|\, E \subseteq U \times U \right\}$$

where $U \times U$ is the set of unordered pairs of distinct elements of U. For any $\sigma \in Hom(U, U')$, the transport of $\mathcal{G}$-structure of σ is actually relabelling vertices using σ.

• The species $\mathcal{S}$ of permutations, which assigns each finite set U to the set of bijections from U to itself, i.e. $\mathcal{S}[U] = \{\tau : U \to U \,|\, \tau \text{ is a bijection}\}$. If $\sigma \in Hom(U, U')$, then $\mathcal{S}[\tau] = \sigma \circ \tau \circ \sigma^{-1}$. A permutation τ can be visualized as a directed graph with vertex set U and have arrows from u to $\tau(u)$.

• The species $\mathcal{C}$ of cyclic permutations, which defined similar as above.

• The species $\mathcal{L}$ of linear orders, which assigns each finite set U to the set of all ways to turn U into an ordered list. Then

$$\mathcal{L}[U] = \{(x_1, \cdots, x_n) \,|\, n = |U|, x_i \in U, \forall i\}$$

The transport of $\mathcal{L}$-structure along σ is

$$\mathcal{L}[\sigma](x_1, \cdots, x_n) = \left(\sigma(x_1), \cdots, \sigma(x_n)\right)$$

A linear order $(x_1, \cdots, x_n)$ can also be visualized as a directed graph with arrows from x_i to x_{i+1}

3. Associated Series

There are three important power series related to the enumeration of F-structures. The basic one is (exponential) generating series, which is related to labelled structures. Then the type generating series is for unlabelled structures as unlabelled structures is an isomorphism class of F-structures. And we have the cycle index series as a general enumeration tool. For the following section, use $[n]$ to denote the set $\{1, \cdot, n\}$ and $F[n]$ to denote the set $F[\{1, \cdots, n\}]$.

Definition 3.1. *The (exponential) generating series of a labelled structure F is the formal power series*

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

where $f_n = |F[n]|$ is the the number of F-structures on a finite set with n elements.

We denote the n-th coefficient of a power series to be $[x^n]$. For an ordinary power series $G(x) = \sum_{n\geq 0} g_n x^n$, have $[x^n]G(x) = g_n$. So for the above exponential power series, have $f_n = n![x^n]F(x) = \frac{d^n F(x)}{dx^n}|_{x=0}$ by Taylor expansion. More generally, if more variables are involved, i.e.

$$H(x_1, x_2, \cdots) = \sum_{n_1, n_2, \cdots} h_{x_1, x_2, \cdots} \frac{x_1^{n_1} x_2^{n_2} \cdots}{c_{n_1, n_2, \cdots}}$$
$$c_{n_1, n_2, \cdots}[x_1^{n_1} x_2^{n_2} \cdots] H(x_1, x_2, \cdots) = h_{x_1, x_2, \cdots}$$

where $c_{n_1, n_2, \cdots}$ is a family of non-zero scalars, then we have the general equation:

We look at the species $\mathcal{L}$ of linear orderings on set U of size n. There are $n!$ ways to arrange these n elements. So the generating series for $\mathcal{L}$ is

$$L(x) = \sum_{n\geq 0} n! \frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$

With similar approach, we have the following corresponding generating series:

Species	Generating Series	Species	Generating Series
$\mathbf{0}$	$\mathbf{0}$(x)=0	$\mathcal{G}$	$\mathcal{G}(x) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$
$\mathbf{E}$	$\mathbf{E}$(x)=e^x	$\mathcal{S}$	$\mathcal{S}(x) = \frac{1}{1-x}$
$\mathbf{1}$	$\mathbf{1}$(x)=1	$\mathcal{C}$	$\mathcal{C}(x) = -\ln(1-x)$
$\mathbf{X}$	$\mathbf{X}$(x)=x	$\mathcal{L}$	$\mathcal{L}(x) = \frac{1}{1-x}$
$\mathbf{E_m}$	$\mathbf{E_m}$(x)=x^m		

Now let's consider the enumeration of unlabelled $\mathcal{F}$-structures. This can be done via enumeration of isomorphism types of $\mathcal{F}$-structures since unlabelled structures can be viewed as isomorphism classes of labelled structures under label permutations. For example, here are 4 unlabelled $\mathcal{G}$-structures where $|U_i| = 3$:

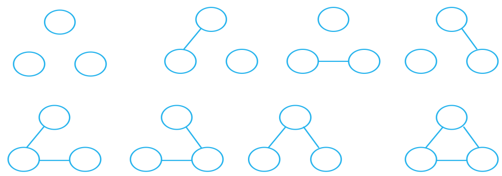

Figure 4: Example of Four Unlabelled Structure Classes

Definition 3.2. *We define an equivalence relation ~ on the set* $F[n]$*, where* $U = [n]$*, for any* $s, t \in F[n]$*:*

$$s \sim t \text{ if and only if } s \text{ and } t \text{ have the same isomorphism type}$$

In other words, $s \sim t$ *if and only if there exists a permutation* $\pi : [n] \to [n]$ *such that* $F[\pi](s) = t$

It is straightforward to see that this relation is indeed an equivalence relation. By definition, one can check that an isomorphism type of F-structures of order n is an equivalence class of F-structures on $[n]$.

Definition 3.3. *The equivalent class of* F*-structures on* $[n]$ *is called an unlabelled* F*-structure of order* n*. We denote*

$$\begin{aligned} T(F_n) &= F[n]/\sim \\ T(F) &= \sum_{n\geq 0} T(F_n) \end{aligned}$$

Similarly to labelled structures, we define the generating series for unlabelled structures.

Definition 3.4. *The type generating series of species* F *is the formal power series*

$$\tilde{F}(x) = \sum_{n\geq 0} \tilde{f}_n x^n$$

where $\tilde{f}_n = |T(F_n)|$ *is the number of unlabelled structures of order* n*.*

For example, consider the species $\mathcal{S}$ of permutation of n elements. For any $s, t \in \mathcal{S}$ to be equivalent, the lengths of disjoint cycles for s decomposition have to match the lengths of the cycles of t. We can conclude $\tilde{f}_n$ is the number of partitions of the number n, thus $\tilde{\mathcal{S}}(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$. We could also compute $\tilde{\mathcal{L}}(x) = \frac{1}{1-x} \neq \tilde{\mathcal{S}}(x)$. This actually shows that $\mathcal{L}$ and $\mathcal{S}$ are different species although $\mathcal{L}(x) = \mathcal{S}(x) = \frac{1}{1-x}$, which is also the reason why we need type generating series.

Definition 3.5. *The cycle index series of a species of a species of structures* F *is the formal power series*

$$Z_F(x_1, x_2, \cdots) = \sum_{n\geq 0} \frac{1}{n!}\Big(\sum_{\sigma \in S_n} (fix\, F[\sigma]) x_\sigma \Big)$$

where S_n *is the permutation group pf* $[n]$*, fix* $F[\sigma] = (F[\sigma])_1$ *is the number of* F*-structures on* $[n]$ *fixed by* $F[\sigma]$ *and* $x_\sigma = x_1^{\sigma_1} x_2^{\sigma_2} \cdots$*.*

By definition, the cycle index series is a generalization of both generating series and the type generating series. We have the following theorem

Theorem 3.6. *For any species of structures F:*

$$F(x) = Z_F(x, 0, 0, \cdots)$$
$$\tilde{F}(x) = Z_F(x, x^2, x^3, \cdots)$$

Sometimes it is difficult to directly compute the type generating series as the structural symmetries may be involved. With the cycle generating series, this problem can be solved in some cases.

We could see with Theorem 3.6, for the species $\mathcal{S}$ of permutations,we have

$$\mathcal{S}(x) = Z_{\mathcal{S}}(x, 0, 0, \cdots) = \frac{1}{1-x}$$
$$\tilde{\mathcal{S}}(x) = Z_{\mathcal{S}}(x, x^2, x^3, \cdots) = \frac{1}{(1-x)(1-x^2)\cdots}$$

4. Category of Species

As mentioned before, isomorphism plays a very important role in the analysis of species, so it is natural to think about the isomorphisms between two species. From the perspective of category theory, it is natural to think about the natural transformation after knowing species is a functor.

Definition 4.1. *A natural transformation α of species F and G assigns each finite set U a function $\alpha_U : F[U] \to G[U]$, in such a way that for any bijection $\sigma : U \to U'$, the following diagram commutes:*

$$\begin{tikzcd} F[U] \arrow[r, "{F[\sigma]}"'] \arrow[d, "\alpha_U"'] & F[U'] \arrow[d, "\alpha_{U'}"] \\ G[U] \arrow[r, "{G[\sigma]}"] & G[U'] \end{tikzcd}$$

If each α_U is a bijection, we say α_U is a natural equivalence of species and F and G are naturally equivalent, i.e. isomorphic. We write $F \simeq G$

Theorem 4.2. *Naturally equivalent species have the same generating series. But the converse is not always true ($\mathcal{L} \not\simeq \mathcal{S}$ is a counter-example).*

Now recall from Assignment 1, we can make a functor category of functors between two categories. Consider the category $\mathcal{F}$ in **Theorem 2.2** and the fact that a species if a functor, also with the natural transformation defined in **Definition 3.7** we can have the following theorem.

Theorem 4.3. *The collection of species forms a category, say $\mathcal{S}pe$, where object of the category are species and morphisms between species are the natural transformations defined above.*

Recall from **Definition 2.3**, a species may related to a symmetric sequence in finite sets. We consider the permutation category $\mathbb{P}$, where $Ob(\mathcal{P})$ consists of natural number and the morphism $f : n \to m$ is defined as

$$f : n \to m = \begin{cases} \sigma \in S_n & \text{if } n = m \\ \varnothing & \text{otherwise} \end{cases}$$

which is equivalent to the category of finite sets and bijections. Then we consider the following definition

Definition 4.4. *A symmetric sequence is a sequence of objects where n-th object has an action of S_n, the $n-$th symmetric group. Moreover it correspond to a category, denoted as $\mathcal{S}ym$ (of finite sets).*

With above construction, one could see that

Theorem 4.5. *The category $\mathcal{S}pe$ is equivalent to the category $\mathcal{S}ym$*

So one can analysis the category $\mathcal{S}ym$ to understand $\mathcal{S}pe$. Furthermore, it is highly related to monoidal structure, which commutes with colimits.

5. Interspecies Operations

There are several basic operations on species

Definition 5.1 (Sum). *If $F_1, F_2, \cdots \in Ob(\mathcal{S}pe)$ and U a finite set, we define*

$$\Big(\bigoplus_{i \geq 0} F_i\Big)[U] = (\{1\} \times F_1)[U] \cup (\{2\} \times F_2)[U] \cup \cdots$$

We write $\{i\} \times F_i$ instead of F_i to make sure the union is disjoint.

Definition 5.2 (Product). *If $F, G \in Ob(\mathcal{S}pe)$, then and $(F \cdot G)$-structure on U is a pair, consisting of an F-structure on X and a G-structure on $U \setminus X$, where $X \subset U$, i.e.*

$$(F \cdot G)[U] = \dot{\bigcup_{X \subset U}} F[X] \times G[U \setminus X]$$

Definition 5.3 (Marking). *If $F \in Ob(\mathcal{S}pe)$, then $F^{\bullet}$-structure on U is a pair (σ, u), where σ is an F-structure on U and $u \in U$ is the marked element, i.e.*

$$F^{\bullet}[U] = F[U] \times U$$

For example, $\mathcal{T}^{\bullet}$ is the species of rooted trees, where the root is the marked element.

Definition 5.4 (Composition). *If $F, G \in Ob(\mathcal{S}pe)$ G is connected (i.e.$G[\varnothing] = \varnothing$). an $F \circ G$-structure on U consists of a set partition of P on U, an F-structure on P and G-functions on P, where a G-function is a function g such that $g(S) \in F[S], \forall S \in P$, i.e.*

$$F \circ G[U] = \dot{\bigcup_{P \text{ is a partition of } U}} F[U] \times \{G - functions \ on \ P\}$$

Theorem 5.5. *With above species $F, G \in Ob(\mathcal{S}pe)$, have*

- $(F+G)(x) = F(x) + G(x) \quad \widetilde{(F+G)}(x) = \tilde{F}(x) + \tilde{G}(x)$
  $Z_{F+G}(x_1, \cdots) = Z_F(x_1, \cdots) + Z_G(x_1, \cdots)$
- $(F \cdot G)(x) = F(x) \cdot G(x) \quad \widetilde{(F \cdot G)}(x) = \tilde{F}(x) \cdot \tilde{G}(x)$
  $Z_{F \cdot G}(x_1, \cdots) = Z_F(x_1, \cdots) \cdot Z_G(x_1, \cdots)$
- $F^{\bullet}(x) = \dfrac{d}{dx} F(x) \quad \widetilde{F^{\bullet}}(x) = (\dfrac{\partial}{\partial x_1} Z_F)(x, x^2, \cdots)$
- $(F \circ G)(x) = F\big(G(x)\big) \quad \widetilde{(F \circ G)}(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \cdots)$

Proof is easily checked by definition.

Now we have addition and multiplication but how about their inverses? Then we consider the concept of virtual species, which is similar to the definition of $\mathbb{Z}$ in terms of $\mathbb{N}$. With this concept, we can define division.

6. Virtual Species

Definition 6.1. *Let Sp denote the subcategory of unisort and noweighted species. We define the virtual species $F - G$ as the element (F, G) in $Ob(Sp) \times Ob(Sp)/\sim$, where $\sim$ is the equivalence relation*

$$(F, G) \sim (A, B) \text{ if and only if } F + B \simeq G + A$$

The additive inverse of F, denoted by $-F$, is given as $-F = \mathbf{0} - F$

Addition, multiplication, marking and composition work on virtual species as they do on regular species. Similarly as $\mathbb{Z}$ being a ring, we can also see that the set of virtual species forms a ring.

Definition 6.2. *The set of virtual species forms a commutative ring S under addition and multiplication defined as*

$$(F - G) + (A - B) = (F + A) - (G + B)$$
$$(F - G) \cdot (A - B) = (F \cdot A + G \cdot B) - (F \cdot B + G \cdot A)$$

where $0_S = (0 - 0)$, $1_S = (1 - 0)$. The additive inverse of $(F - G)$ is $(G - F)$. Moreover, the injection $Sp \to S$ via $F \mapsto (F - 0)$ is a semi-ring homomorphism.

With above construction, we could write any virtual species $F = F^+ - F^-$, where the only subspecies of F^+ which is isomorphic to a subspecies of F^- is the empty species. It is easy to verify that there does not exist a species G of same structure such that $F \cdot G = \mathbf{1}$. But virtual species allows us to define the multiplicative inverse.

Definition 6.3 (Division). *Given a species F with one structure on the empty set, we define*

$$F^{-1} = \frac{1}{1 + F_+} = \sum_{n=0}^{\infty} (-1)^n (F_+^n)$$

where the family $\{F_+^n\}$ is summable, i.e. $\forall U$, $\exists N \in \mathbb{N}$ s.t. $F_+^n[U] = \varnothing \ \forall n \geq N$.

Theorem 6.4. *With simple computation, we can have*

- $(F - G)(x) = F(x) - G(x) \quad \widetilde{F - G}(x) = \tilde{F}(x) - \tilde{G}(x)$
  $Z_{F-G}(x_1, x_2, \cdots) = Z_F(x_1, x_2, \cdots) - Z_G(x_1, x_2, \cdots)$
- $F^{-1}(x) = e^{-x}$; $\widetilde{F^{-1}}(x) = 1 - x$; $Z_{F^{-1}}(x_1, x_2, \cdots) = e^{-\sum_{i \geq 1} x_i / i}$

7. Furthermore

As an undergrad, it may be difficult to understand the theory of species totally as it involves many prerequisites. But I can still see the great influence of category theory when developing species as it serves as a bridge between algebra and combinatorics. Further topics relates to HoTT (Homotopy Type Theory) and monoidal structures etc.

References

[1] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial Species and Tree-like Structures.* Cambridge University Press, 1997.

[2] Brent Abraham Yorgey. *Combinatorial Species and Labelled Structures (Ph.D. thesis).* 2014.

[3] A. Hardt, P. Mcneely, T. Phan, and J.M. Troyka. *Combinatorial Species and Graph Enumeration.* https://arxiv.org/pdf/1312.0542.pdf

[4] André Joyal. "Une th´eorie combinatoire des s´eries formelles". *Advances in Mathematics*, 42 (1981). 1–82.

[5] M. Aguiar and S.Mahajan. *Hopf monoids in the category of species.* https://arxiv.org/pdf/1210.3120.pdf